Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2017

Calculus

Higher level

Paper 3

12 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. attempt to use l'Hôpital's rule, ***M1***

limit $= \lim_{x \to 0} \dfrac{2\sin x \cos x}{\ln(1+x) + \dfrac{x}{1+x}}$ or $\dfrac{\sin 2x}{\ln(1+x) + \dfrac{x}{1+x}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

this gives $0/0$ so use the rule again ***(M1)***

$= \lim_{x \to 0} \dfrac{2\cos^2 x - 2\sin^2 x}{\dfrac{1}{1+x} + \dfrac{1+x-x}{(1+x)^2}}$ or $\dfrac{2\cos 2x}{\dfrac{2+x}{(1+x)^2}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

$= 1$ ***A1***

Note: This ***A1*** is dependent on all previous marks being awarded, except when the first application of L'Hopital's does not lead to $0/0$, when it should be awarded for the correct limit of their derived function.

[7 marks]

2. (a) (i) $(\sec^2 x =)\ a_1 + 3a_3x^2 + 5a_5x^4 + \dots$ ***A1***

(ii) $\sec^2 x = 1 + \left(a_1x + a_3x^3 + a_5x^5 + \dots\right)^2$

$= 1 + a_1^2x^2 + 2a_1a_3x^4 + \dots$ ***M1A1***

Note: Condone the presence of terms with powers greater than four.

[3 marks]

(b) equating constant terms: $a_1 = 1$ ***A1***

equating x^2 terms: $3a_3 = a_1^2 = 1 \Rightarrow a_3 = \dfrac{1}{3}$ ***A1***

equating x^4 terms: $5a_5 = 2a_1a_3 = \dfrac{2}{3} \Rightarrow a_5 = \dfrac{2}{15}$ ***A1***

[3 marks]

Total [6 marks]

3. consider $I = \int_2^N \frac{\mathrm{d}x}{x\sqrt{\ln x}}$ ***M1A1***

Note:	Do not award ***A1*** if n is used as the variable or if lower limit equal to 1, but some subsequent ***A*** marks can still be awarded. Allow ∞ as upper limit.

let $y = \ln x$ ***(M1)***

$\mathrm{d}y = \frac{\mathrm{d}x}{x}$, ***(A1)***

$[2, N] \Rightarrow [\ln 2, \ln N]$

$I = \int_{\ln 2}^{\ln N} \frac{\mathrm{d}y}{\sqrt{y}}$ ***(A1)***

Note:	Condone absence of limits, or wrong limits.

$= \left[2\sqrt{y}\right]_{\ln 2}^{\ln N}$ ***A1***

Note:	***A1*** is for the correct integral, irrespective of the limits used. Accept correct use of integration by parts.

$= 2\sqrt{\ln N} - 2\sqrt{\ln 2}$ ***(M1)***

Note:	***M1*** is for substituting their limits into their integral and subtracting.

$\to \infty$ as $N \to \infty$ ***A1***

Notes:	Allow "$= \infty$", "limit does not exist", "diverges" or equivalent. Do not award if wrong limits substituted into the integral but allow N or ∞ as an upper limit in place of $\ln N$.

(by the integral test) the series is divergent (because the integral is divergent) ***A1***

Notes:	Do not award this mark if ∞ used as upper limit throughout.

[9 marks]

4. (a) $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***M1***

the differential equation becomes

$v + x\frac{dv}{dx} = f(v)$ ***A1***

$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$ ***A1***

integrating, $\int \frac{dv}{f(v) - v} = \ln x + \text{Constant}$ ***AG***

[3 marks]

(b) **EITHER**

$f(v) = 1 + 3v + v^2$ ***(A1)***

$\left(\int \frac{dv}{f(v) - v} = \right)\int \frac{dv}{1 + 3v + v^2 - v} = \ln x + C$ ***M1A1***

$\int \frac{dv}{(1+v)^2} = (\ln x + C)$ ***A1***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v}(= \ln x + C)$ ***A1***

OR

$v + x\frac{dv}{dx} = 1 + 3v + v^2$ ***A1***

$\int \frac{dv}{1 + 2v + v^2} = \int \frac{1}{x}dx$ ***M1***

$\int \frac{dv}{(1+v)^2}\left(= \int \frac{1}{x}dx\right)$ ***(A1)***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v} = \ln x(+C)$ ***A1A1***

continued…

Question 4 continued

THEN

substitute $y = 1$ or $v = 1$ when $x = 1$ ***(M1)***

therefore $C = -\frac{1}{2}$ ***A1***

Note: This ***A1*** can be awarded anywhere in their solution.

substituting for v,

$$-\frac{1}{\left(1+\frac{y}{x}\right)} = \ln x - \frac{1}{2}$$ ***M1***

Note: Award for correct substitution of $\frac{y}{x}$ into their expression.

$$1+\frac{y}{x} = \frac{1}{\frac{1}{2} - \ln x}$$ ***(A1)***

Note: Award for any rearrangement of a correct expression that has y in the numerator.

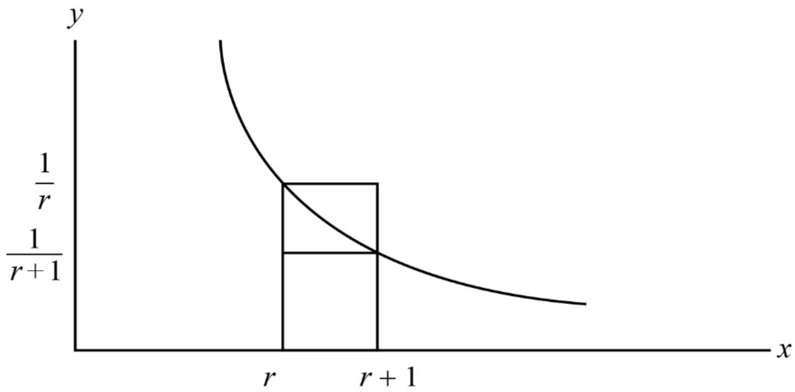
$$y = x\left(\frac{1}{\left(\frac{1}{2} - \ln x\right)} - 1\right)$$ (or equivalent) ***A1***

$$\left(= x\left(\frac{1 + 2\ln x}{1 - 2\ln x}\right)\right)$$

[10 marks]

Total [13 marks]

5. (a)



A1

Note: Curve, both rectangles and correct x- values required.

area of rectangles $\frac{1}{r}$ and $\frac{1}{1+r}$ ***A1***

Note: Correct values on the y-axis are sufficient evidence for this mark if not otherwise indicated.

in the above diagram, the area below the curve between $x = r$ and $x = r + 1$ is between the areas of the larger and smaller rectangle

or $\frac{1}{r+1} < \int_r^{r+1} \frac{dx}{x} < \frac{1}{r}$ ***(R1)***

integrating, $\int_r^{r+1} \frac{dx}{x} = [\ln x]_r^{r+1} \left(= \ln(r+1) - \ln(r)\right)$ ***A1***

$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}$ ***AG***

[4 marks]

(b) (i) summing the right-hand part of the above inequality from $r = 1$ to $r = n$,

$\sum_{r=1}^{n} \frac{1}{r} > \sum_{r=1}^{n} \ln\left(\frac{r+1}{r}\right)$ ***M1***

$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right)$ ***(A1)***

EITHER

$= \ln\left(\frac{2}{1} \times \frac{3}{2} \times \ldots \times \frac{n}{n-1} \times \frac{n+1}{n}\right)$ ***A1***

OR

$\ln 2 - \ln 1 + \ln 3 - \ln 2 + \ldots + \ln(n+1) - \ln(n)$ ***A1***

$= \ln(n+1)$ ***AG***

continued…

Question 5 continued

(ii) $\sum_{r=1}^{n}\frac{1}{r}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<1+\ln\left(\frac{2}{1}\right)+\ln\left(\frac{3}{2}\right)+\ldots+\ln\left(\frac{n}{n-1}\right)$ ***M1A1A1***

$\left(1+\sum_{r=1}^{n-1}\frac{1}{r+1}<1+\sum_{r=1}^{n-1}\ln\left(\frac{r+1}{r}\right)\right)$

Note:	***M1*** is for using the correct inequality from (a), ***A1*** for both sides beginning with 1, ***A1*** for completely correct expression.

Note:	The 1 might be added after the sums have been calculated.

$=1+\ln n$ ***AG***

[6 marks]

(c) (i) from (b)(i) $U_n > \ln(1+n) - \ln n > 0$ ***A1***

(ii) $U_{n+1} - U_n = \sum_{r=1}^{n+1}\frac{1}{r} - \ln(n+1) - \sum_{r=1}^{n}\frac{1}{r} + \ln n$ ***M1***

$=\frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right)$ ***A1***

< 0 (using the result proved in (a)) ***A1***

$U_{n+1} < U_n$ ***AG***

[4 marks]

(d) it follows from the two results that $\{U_n\}$ cannot be divergent either in the sense of tending to $-\infty$ or oscillating therefore it must be convergent ***R1***

[1 mark]

Note:	Accept the use of the result that a bounded (monotonically) decreasing sequence is convergent (allow "positive, decreasing sequence").

Total [15 marks]